

APPROXIMATELY FINITE DIMENSIONAL C^* -ALGEBRAS AND BRATTOLI DIAGRAMS

BY

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ABSTRACT. We determine properties of an AF algebra by observing the characteristics of its diagram. In particular, we characterize AF algebras that are liminal, postliminal, antiliminal and with continuous trace; moreover, we characterize liminal AF algebras with Hausdorff spectrum. Some elementary examples of AF algebras with certain desired properties are constructed by using these characterizations.

1. Introduction. An approximately finite dimensional C^* -algebra, or, in short, an AF algebra, is defined to be the norm closure of an ascending sequence of finite dimensional C^* -algebras. There is considerable evidence that AF algebras serve as models for much more general and important algebras. Consequently, AF algebras have attracted a lot of study since their introduction by Bratteli [1]. By using diagrams, a device developed by Bratteli [1], AF algebras become relatively simple to handle without being trivial. They are especially well suited to test conjectures and to provide examples in the theory of C^* -algebras.

The purpose of this paper is to explore further the use of diagrams in the study of AF algebras. In §2 we give a variant of the basic results on AF algebras and diagrams developed by Bratteli. In Bratteli's original work, an AF algebra was assumed to have an identity. We do not make this a requirement. In §3 we develop some new results that allow us to determine properties of an AF algebra by observing the characteristics of its diagram. In particular, we characterize AF algebras that are liminal, postliminal, antiliminal, and with continuous trace; moreover, we characterize liminal AF algebras with Hausdorff spectrum. In §4, we construct some elementary examples of AF algebras with certain desired properties by using the results of §3. Other examples with these desired properties already exist. However, they are much more complicated. In one case the example was a solution to a problem of Dixmier that was open for seventeen years. Our examples clearly demonstrate the power and usefulness of diagrams in the construction of examples.

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2. Approximately finite dimensional C^* -algebras and Bratteli diagrams—basic results. Let A be a C^* -algebra. The C^* -algebra is said to be *approximately finite dimensional* if there exists an increasing (with respect to inclusion) sequence $\{A_n\}_{n=1}^\infty$ of finite dimensional $*$ -subalgebras of A such that A is the uniform closure of $\bigcup A_n$, that is, $A = \overline{\bigcup A_n}$. Although A may have an identity, we do not make this condition a requirement. We sometimes refer to approximately finite dimensional C^* -algebras as *AF algebras*. For some properties of an AF algebra, the adjoining of an identity does not interfere with them (in fact, it may simplify things), but for other properties it does. For example, the algebra obtained by adjoining an identity to a liminal AF algebra may not be liminal. In this section we present some results that allow us to connect AF algebras with their *diagrams*, which we define in the paragraph below. We would like to point out that all of the work in this section is a variant of Bratteli's work on AF algebras with identity [1].

Let $D = \{(i, m)\}_{m=1, i=1}^{\infty, r(m)}$, where $\{r(m)\}_{m=1}^\infty$ is a sequence of positive integers. For each positive integer m let U_m be an $r(m)$ by $r(m+1)$ matrix with nonnegative integers as entries and such that each row of U_m is nonzero. Let \mathcal{U} denote the sequence of matrices $\{U_m\}_{m=1}^\infty$. Next let $d: D \rightarrow N$, where N denotes the natural numbers, be a map such that

$$d((j, m+1)) \geq \max \left\{ 1, \sum_{i=1}^{r(m)} \beta_{i,j} d((i, m)) \right\}, \quad 1 \leq j \leq r(m+1), \quad (1)$$

where $\beta_{i,j}$ denotes the entry of the i th row of the j th column of the matrix U_m . The ordered triple (D, d, \mathcal{U}) is called a *modified Bratteli diagram*, or in short, a *diagram*. The map d is called the *dimensional function* of the diagram and we say $d((i, m))$ is the dimension of the point (i, m) . Let us consider some specific examples of diagrams.

2.1. EXAMPLE. Let $\{r(m)\}_{m=1}^\infty$ be defined by the formula

$$r(m) = \begin{cases} 2, & m = 1, 2, \\ 3, & m > 2, \end{cases}$$

and set $D = \{(i, m)\}_{m=1, i=1}^{\infty, r(m)}$. Let

$$U_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}, \quad U_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for $m > 3$. Let

$$d((i, m)) = \begin{cases} 1, & i = 1, m = 1, 2, \dots, \\ m, & i = 2, m = 1, 2, \dots, \\ 2, & i = 3, m = 3, \\ 10, & i = 3, m > 3. \end{cases}$$

2.2. EXAMPLE. Let $\{r(m)\}_{m=1}^\infty$ be defined by the formula

$$r(m) = \begin{cases} 1, & 1 \leq m \leq 5, \\ 2, & 5 < m, \end{cases}$$

and set $D = \{(i, m)\}_{m=1, i=1}^{\infty, r(m)}$. Let $U_m = [1]$, $1 \leq m \leq 4$, $U_5 = [1, 2]$, and $U_m = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $m > 5$. Let

$$d((i, m)) = \begin{cases} m, & i = 1, m \geq 1, \\ 10, & i = 2, m \geq 5. \end{cases}$$

It is more illuminating to see the associated graphs of the above diagrams, which we present below. The number in parentheses by the point on the graph corresponds to the dimension of the point. If the i th row of the j th column of the matrix U_m is nonzero, then there is an edge (segment) connecting the point (i, m) to the point $(j, m + 1)$. If it is greater than 1, then that number is written by the edge connecting the two points. If it is 1, then no number is written by the edge, but one should understand that 1 could be written there. If it is zero, there is no edge connecting the two points. The number by the edge is called the multiplicity of the edge. The dotted lines connecting the two graphs will be explained later. It is clear that every diagram has a graph and, moreover, every graph with the above properties defines a diagram. Since graphs are more illuminating, all future examples of diagrams will be presented in their graphical form.

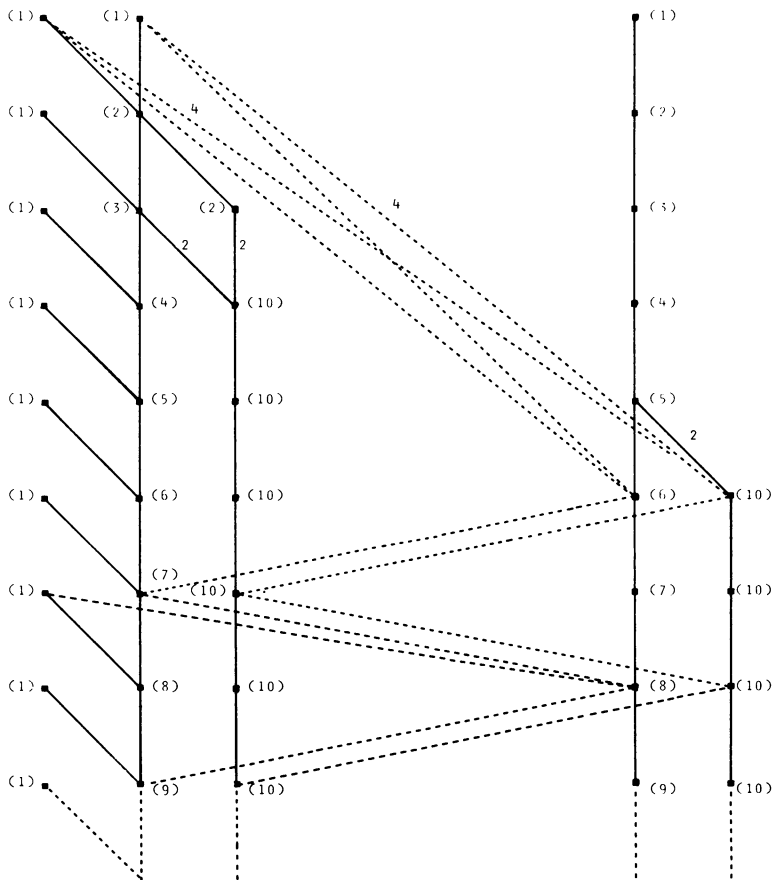


FIGURE 1. Examples 2.1 and 2.2

Now let (D, d, \mathcal{U}) be a diagram, where $D = \{(i, m)\}_{m=1, i=1}^{\infty, r(m)}$ and $\mathcal{U} = \{U_m\}_{m=1}^{\infty}$. Let (i, m) and (j, n) belong to D with $n > m$. The point (j, n) is said to be a *descendant*, with multiplicity q , of the point (i, m) if q is nonzero and it is the entry of the i th row of the j th column of the matrix U_m , $n = m + 1$, or of the matrix $U_m U_{m+1} \dots U_{n-1}$, $n > m + 1$. The number q corresponds to the number of paths from (i, m) to (j, n) , where an edge with multiplicity p is viewed as generating p paths. For example, consider the points $(1, 1)$ and $(3, 4)$ of Example 2.1. The point $(3, 4)$ is a descendant of $(1, 1)$ with multiplicity 4. Sometimes we may just say that a point (j, n) is a *descendant* of a point (i, m) or that (i, m) is an *ancestor* of (j, n) , or in short, $(j, n) < (i, m)$ or $(i, m) > (j, n)$. This relation defines a partial ordering on D . For fixed m , the set

$$D_m = \{(i, m): (i, m) \in D\}$$

will be called the m th *generation* of the diagram. Next, let $\{x_k\}_{k=1}^{\infty}$ a sequence of points in D with the following property: x_{k+1} is a descendant of x_k , $k \geq 1$. Any such sequence in D is called a *connected sequence*. If for every $k \geq 1$ x_{k+1} is at the next generation following the generation to which x_k belongs, then $\{x_k\}$ is called a *complete connected sequence*. Let $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty}$ be connected sequences in D with x_j and y_j at the same generation, $j \geq 1$. The two sequences are said to be *eventually the same* if there is a positive integer q such that $x_j = y_j$, $j \geq q$.

Now let (D', d', \mathcal{U}') be another diagram, where $D' = \{(j, n)\}_{n=1, j=1}^{\infty, s(n)}$ and $\mathcal{U}' = \{U'_n\}$. The diagram (D', d', \mathcal{U}') is said to be *equivalent* to the diagram (D, d, \mathcal{U}) if and only if there exist a diagram (E, e, \mathcal{V}) , where $E = \{(i, n)\}_{n=1, i=1}^{\infty, t(n)}$ and $\mathcal{V} = \{V_n\}_{n=1}^{\infty}$, and an increasing sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ such that the following hold:

- (i) $t(n) = r(m_n)$ and $V_n V_{n+1} = U_{m_n} U_{m_n+1} \dots U_{m_{n+2}-1}$, n an odd integer;
- (ii) $t(n) = s(m_n)$ and $V_n V_{n+1} = U'_{m_n} U'_{m_n+1} \dots U'_{m_{n+2}-1}$, n an even integer;
- (iii) $e((i, n)) = d((i, m_n))$, n an odd integer, and $e((i, n)) = d'((i, m_n))$, n an even integer.

It is straightforward to show that the above defines an equivalence relation. The diagrams given in Examples 2.1 and 2.2 are equivalent. The dotted lines connecting the two diagrams indicate how a diagram (E, e, \mathcal{V}) can be chosen. More specifically, let (E, e, \mathcal{V}) be the diagram given below (Figure 2).

Let $(D, d, \mathcal{U}) = (\{(i, n)\}_{n=1, i=1}^{\infty, r(n)}, d, \{U_n\}_{n=1}^{\infty})$ be a diagram. A diagram $(E, e, \mathcal{V}) = (\{(i, n)\}_{n=1, i=1}^{\infty, s(n)}, e, \{V_n\}_{n=1}^{\infty})$ is called a *rearrangement* of the diagram (D, d, \mathcal{U}) if and only if the following conditions are satisfied: (i) $r(n) = s(n)$, $n = 1, 2, \dots$; (ii) there exists a map $T: E \rightarrow D$ that is one-to-one, onto, and such that $T(E_n) = D_n$, where

$$D_n = \{(i, n) \in D: 1 \leq i \leq r(n)\} = \{(i, n) \in E: 1 \leq i \leq s(n)\} = E_n;$$

- (iii) if $T((i, n)) = (p, n)$ and $T((j, n+1)) = (q, n+1)$, then the entry of the i th row and j th column of the matrix V_n is equal to the entry of the p th row and q th column of the matrix U_n . Note that any triple (E, e, \mathcal{V}) that satisfies the above is a diagram (Figure 3).

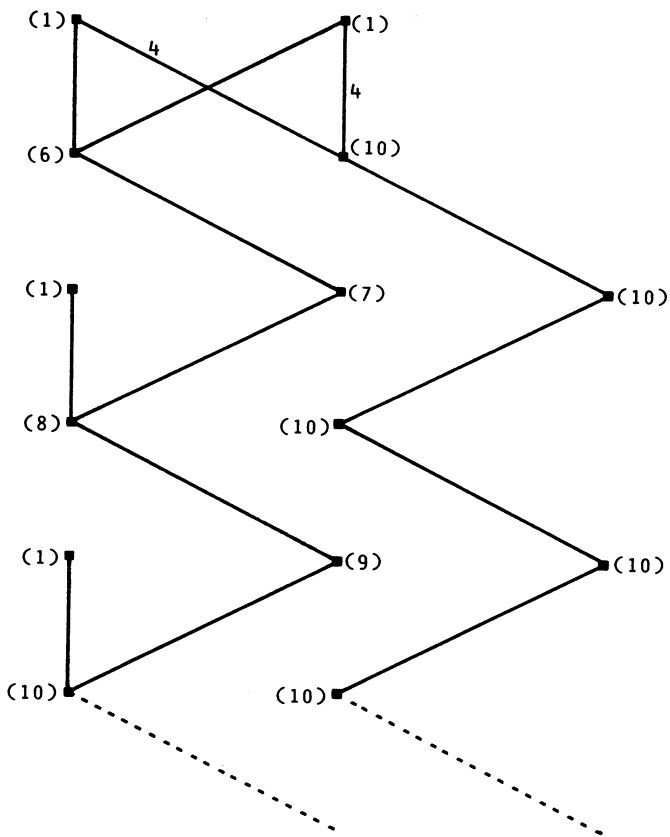


FIGURE 2. Example 2.3

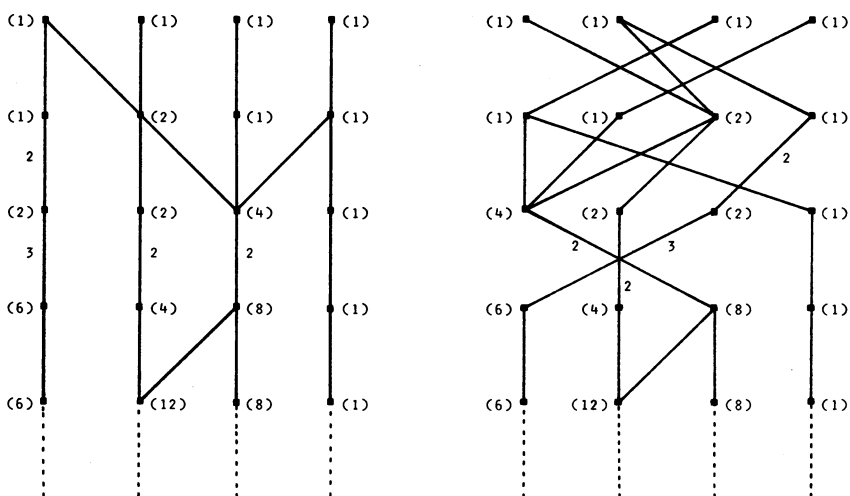


FIGURE 3. Example 2.4

The example above is of a diagram and one of its rearrangements. Although a rearrangement of a diagram may appear to be quite different from the original diagram, the following lemma explains how they are related.

2.5. LEMMA. *Let (D, d, \mathcal{U}) be a diagram and let (E, e, \mathcal{V}) be a rearrangement of (D, d, \mathcal{U}) . Then (E, e, \mathcal{V}) is equivalent to (D, d, \mathcal{U}) .*

PROOF. The proof is straightforward.

Of course, it is easy to see that the above definitions and concepts concerning diagrams can be extended to include all infinite subsets D of $N \times N$ with the property that for each $n \in N$, $\{(m, n): (m, n) \in D\}$ is finite (possibly empty). Consequently, throughout the rest of this paper we shall view diagrams in their general setting.

Now let $(D, d, \mathcal{U}) = (\{(i, n)\}_{n=1, i=1}^{\infty, r(m)}, d, \{U_n\}_{n=1}^{\infty})$ be a diagram and let $E = \{(j_{p,k}, n_p)\}_{p=1, k=1}^{\infty, s(p)}$ be an infinite subset of D , where $j_{p,1} < j_{p,2} < \dots < j_{p,s(p)}$, and such that each element of E has a descendant that belongs to E . For each positive integer p let V_p be the $s(p) \times s(p+1)$ matrix, where the entry of the i th row and q th column of V_p is equal to the entry of the $j_{p,i}$ th row and $j_{p+1,q}$ th column of the matrix W_p , where $W_p = U_{n_p}$ if $n_{p+1} = n_p + 1$ or $W_p = U_{n_p} U_{n_p+1} \dots U_{n_{p+1}-1}$ if $n_{p+1} > n_p + 1$. Set $\mathcal{U}_E = \{V_p\}_{p=1}^{\infty}$. Without much difficulty, one can show that $(E, d|E, \mathcal{U}_E)$ is a diagram. We call $(E, d|E, \mathcal{U}_E)$ a subdiagram of (D, d, \mathcal{U}) and we sometimes refer to $(E, d|E, \mathcal{U}_E)$ as the subdiagram of (D, d, \mathcal{U}) generated by the subset E . If $s(p) = r(n_p)$, for all $p \geq 1$, then $(E, d|E, \mathcal{U}_E)$ is called a subdiagram of (D, d, \mathcal{U}) of the first kind. Here we assumed that the diagram (D, d, \mathcal{U}) was not defined in its general setting in order to simplify the notation, but we certainly could have defined it in the general setting.

2.6. LEMMA. *Let (D, d, \mathcal{U}) be a diagram and let $(E, d|E, \mathcal{U}_E)$ be a subdiagram of (D, d, \mathcal{U}) of the first kind. Then $(E, d|E, \mathcal{U}_E)$ is equivalent to (D, d, \mathcal{U}) .*

PROOF. The proof is straightforward.

2.7. COROLLARY. *Let (D, d, \mathcal{U}) be a diagram and let (E, e, \mathcal{V}) be a rearrangement of a subdiagram of (D, d, \mathcal{U}) of the first kind. Then (E, e, \mathcal{V}) is equivalent to (D, d, \mathcal{U}) .*

PROOF. The proof follows immediately from 2.5 and 2.6.

We will now review some basic results on inductive limits of C^* -algebras and then connect diagrams with AF algebras.

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of C^* -algebras. Suppose that for each positive integer n there is a $*$ -isomorphism ϕ_n of A_n into A_{n+1} . Let \mathcal{L} be the set of all $x = \{x_n\}_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n$ for which there exists a positive integer m such that $x_n = \phi_{n-1}\phi_{n-2} \dots \phi_m(x_m)$ for all $n > m$. Clearly, \mathcal{L} is a $*$ -algebra and $\|x\| \equiv \lim \|x_n\|$ is a C^* -seminorm on \mathcal{L} . Put $\mathcal{I} = \{x \in \mathcal{L}: \|x\| = 0\}$. \mathcal{I} is a selfadjoint two sided ideal. Consider the $*$ -algebra \mathcal{L}/\mathcal{I} . The seminorm $\|\cdot\|$ on \mathcal{L} is constant on each class $y + \mathcal{I}$ ($y \in \mathcal{L}$), and so defines a C^* -norm on \mathcal{L}/\mathcal{I} , also denoted by $\|\cdot\|$. Let $\text{ind } \lim \langle A_n, \phi_n \rangle$ denote the C^* -algebra obtained by the completion of \mathcal{L}/\mathcal{I} . The

C^* -algebra $\text{ind lim} \langle A_n, \phi_n \rangle$ is called the *inductive limit* of $\{A_n\}$ defined by the family of $*$ -isomorphisms $\{\phi_n\}$. Now suppose $\{n_k\}$ is an increasing sequence of positive integers and $B_k = A_{n_k}$. Set $\Psi_k = \phi_{n_{k+1}-1} \phi_{n_{k+1}-2} \dots \phi_{n_k}$. Then it is easy to show that $\text{ind lim} \langle A_n, \phi_n \rangle$ is $*$ -isomorphic to $\text{ind lim} \langle B_k, \Psi_k \rangle$. Next suppose x_k is a fixed element in A_k . Define $y = \{y_n\} \in \prod A_n$ by the formula: $y_n = 0, n < k$; $y_k = x_k, y_n = \phi_{n-1} \phi_{n-2} \dots \phi_k(x_k), n > k$. It is easy to verify that $x_k \rightarrow y + \mathcal{I}$ is a $*$ -isomorphism of A_k into $\text{ind lim} \langle A_n, \phi_n \rangle$. We denote by \tilde{A}_k the image of A_k . Moreover, it is clear that $\tilde{A}_k \subseteq \tilde{A}_{k+1}$ and that $\text{ind lim} \langle A_n, \phi_n \rangle$ is the uniform closure of $\bigcup_{n=1}^{\infty} \tilde{A}_n$. On the other hand if A is a C^* -algebra and $\{A_n\}$ is an increasing sequence of C^* -subalgebras such that A is the uniform closure of $\bigcup A_n$, then A is $*$ -isomorphic to $\text{ind lim} \langle A_n, i_n \rangle$, where i_n is the identity map. If each A_n has an identity and ϕ_n maps the identity of A_n into the identity of A_{n+1} , then all $\{\tilde{A}_n\}_{n=1}^{\infty}$ and $\text{ind lim} \langle A_n, \phi_n \rangle$ have a common identity.

Let \mathcal{Q} denote the family of ordered triples

$$(\{A_n\}_{n=1}^{\infty}, \{F_{i,n}\}_{n=1, i=1}^{\infty, r(n)}, \{\phi_n\}_{n=1}^{\infty})$$

that are defined as follows: (i) $\{A_n\}_{n=1}^{\infty}$ is a sequence of finite dimensional C^* -algebras with nondecreasing dimension; (ii) $\{r(n)\}_{n=1}^{\infty}$ is a sequence of positive integers; (iii) for each ordered pair of positive integers $(i, n), i \leq r(n)$, $F_{i,n}$ denotes a factor contained in A_n , and $A_n = \sum_{i=1}^{r(n)} \oplus F_{i,n}$, the direct sum of the factors $\{F_{i,n}\}_{i=1}^{r(n)}, n = 1, 2, \dots$; (iv) for each positive integer n , ϕ_n is a $*$ -isomorphism of A_n into A_{n+1} . Now let F_1 and F_2 be finite dimensional factors and ϕ a $*$ -homeomorphism of F_1 into F_2 . We define multiplicity of ϕ , denoted by $m(\phi)$, to be equal to $\text{tr}(\phi(q))$, where q denotes a minimal nonzero projection of F_1 . Since any two nonzero minimal projections of F_1 are equivalent, it follows that $m(\phi)$ is well defined. Next, we will show that there is a natural relationship between elements of \mathcal{Q} and diagrams. Let $(\{A_n\}_{n=1}^{\infty}, \{F_{i,n}\}_{n=1, i=1}^{\infty, r(n)}, \{\phi_n\}_{n=1}^{\infty}) \in \mathcal{Q}$ and let $p_{i,n}$ denote the identity projection of $F_{i,n}$. Define $\phi_{n,i,j}: F_{i,n} \rightarrow F_{j,n+1}$ by the formula

$$\phi_{n,i,j}(x) = p_{j,n+1} \phi_n(x), \quad x \in F_{i,n},$$

which clearly is a $*$ -homeomorphism of $F_{i,n}$ into $F_{j,n+1}$. Now let $D = \{(i, n)\}_{n=1, i=1}^{\infty, r(n)}, d((i, n)) = \dim F_{i,n}$, and let $U_n = [\alpha_{i,j}]$ be the $r(n)$ by $r(n+1)$ matrix given by $\alpha_{i,j} = m(\phi_{n,i,j})$. Clearly (D, d, \mathcal{U}) is a diagram, where $\mathcal{U} = \{U_n\}$. The diagram (D, d, \mathcal{U}) is referred to as the canonical diagram generated by

$$(\{A_n\}_{n=1}^{\infty}, \{F_{i,n}\}_{n=1, i=1}^{\infty, r(n)}, \{\phi_n\}_{n=1}^{\infty}).$$

2.8. LEMMA. Let

$$\mathbf{x} = (\{A_n\}_{n=1}^{\infty}, \{E_{i,n}\}_{n=1, i=1}^{\infty, r(n)}, \{\phi_n\}_{n=1}^{\infty})$$

and

$$\mathbf{y} = (\{B_n\}_{n=1}^{\infty}, \{F_{i,n}\}_{n=1, i=1}^{\infty, r(n)}, \{\Psi_n\}_{n=1}^{\infty})$$

belong to \mathcal{Q} and suppose for each positive integer n , λ_n is a $*$ -isomorphism of A_n onto B_n . If $\lambda(E_{i,n}) = F_{i,n}$ and $\Psi_n \lambda_n = \lambda_{n+1} \phi_n, i \leq r(n), i = 1, 2, \dots$, then \mathbf{x} and \mathbf{y} generate the same canonical diagram.

PROOF. The proof is straightforward.

2.9. LEMMA. Let $\mathbf{x} = (\{A_n\}_{n=1}^\infty, \{E_{i,n}\}_{n=1,i=1}^{\infty,r(n)}, \{\phi_n\}_{n=1}^\infty)$ belong to \mathcal{Q} and let $\mathbf{y} = (\{A_{n_k}\}_{k=1}^\infty, \{E_{i,n_k}\}_{k=1,i=1}^{\infty,r(n_k)}, \{\Psi_k\}_{k=1}^\infty)$, where $\{n_k\}$ is a subsequence of the sequence of positive integers and $\Psi_k = \phi_{n_{k+1}-1}\phi_{n_{k+1}-2}\cdots\phi_{n_k}$. Then the canonical diagram generated by \mathbf{y} is a subdiagram of the first kind of the canonical diagram generated by \mathbf{x} , and therefore they are equivalent.

PROOF. The proof is straightforward.

2.10. LEMMA. Let $\mathbf{x} = (\{A_n\}_{n=1}^\infty, \{E_{i,n}\}_{n=1,i=1}^{\infty,r(n)}, \{\phi_n\}_{n=1}^\infty)$ belong to \mathcal{Q} and let $\mathbf{y} = (\{A_n\}_{n=1}^\infty, \{F_{i,n}\}_{n=1,i=1}^{\infty,r(n)}, \{\phi_n\}_{n=1}^\infty)$, where $\{F_{i,n}\}_{i=1}^{r(n)}$ is a rearrangement of $\{E_{i,n}\}_{i=1}^{r(n)}$, $n = 1, 2, \dots$. Then the canonical diagram generated by \mathbf{y} is a rearrangement of the canonical diagram generated by \mathbf{x} and therefore they are equivalent.

PROOF. The proof is straightforward.

Now let A be an AF algebra. We define an associate diagram of A to be a canonical diagram (E, e, ∇) generated by any $(\{A_n\}_{n=1}^\infty, \{F_{i,n}\}_{n=1,i=1}^{\infty,r(n)}, \{i_n\}_{n=1}^\infty) \in \mathcal{Q}$, where $A_n \subseteq A_{n+1} \subseteq A$, i_n is the identity map, and A is the closure of $\bigcup A_n$. Suppose I is a closed two sided ideal of A . Let $K = \{(i, n): F_{i,n} \cap I \neq \{0\}\}$. Note every $x \in K$ has a descendant in K . If K is nonempty, then the subdiagram $(K, e|_K, \nabla_K)$ of (E, e, ∇) is called the canonical ideal subdiagram of (E, e, ∇) associated with the ideal I . The set K is called the canonical ideal subset of D associated with I .

Next let (D, d, \mathcal{U}) be a diagram where $D = \{(i, n)\}_{n=1,i=1}^{\infty,r(n)}$ and $\mathcal{U} = \{U_n\}_{n=1}^\infty$. Let $F_{i,n}$ denote a factor of dimension $d((i, n))$ and set $A_n = \sum_{i=1}^{r(n)} F_{i,n}$. For each positive integer n and positive integers i, j with $i < r(n), j < r(n+1)$ choose a *-isomorphism $\phi_{n,i,j}$ of $F_{i,n}$ into $F_{j,n+1}$ such that the following hold: if $i_1 \neq i_2$ and $i_1, i_2 < r(n)$, then

$$\phi_{n,i_1,j}(x)\phi_{n,i_2,j}(y) = 0$$

for all $x \in F_{i_1,n}, y \in F_{i_2,n}$, and $j = 1, 2, \dots, r(n+1)$; the multiplicity of $\phi_{n,i,j}$ is equal to the entry of the i th row and j th column of U_n . Choose the zero mapping, if the entry is zero. We can choose such a map $\phi_{n,i,j}$ by virtue of (1). Define $\phi_{n,i}: F_{i,n} \rightarrow A_{n+1}$ by the formula

$$\phi_{n,i}(x) = \sum_{j=1}^{r(n+1)} \oplus \phi_{n,i,j}(x)$$

and then define $\phi_n: A_n \rightarrow A_{n+1}$ by the formula

$$\phi_n(x) = \sum_{i=1}^{r(n)} \phi_{n,i}(p_{i,n}x),$$

where $p_{i,n}$ is the identity projection of $F_{i,n}$. It follows from 2.8, that (D, d, \mathcal{U}) is an associate diagram of $\text{ind lim} \langle A_n, \phi_n \rangle$.

2.11. THEOREM. Let A and B be AF algebras and let $\{A_n\}$ and $\{B_n\}$ be increasing sequences of finite dimensional *-subalgebras of A and B , respectively, such that $A = \overline{\bigcup A_n}$ and $B = \overline{\bigcup B_n}$. Then A is *-isomorphic to B if and only if $\{A_n\}$ contains a subsequence $\{A_{n_k}\}$ and each A_{n_k} contains a finite dimensional *-subalgebra B'_k such that:

- (i) $\{B'_k\}$ is an increasing sequence, and there exists a *-isomorphism $\alpha: \bigcup B_k \rightarrow \bigcup B'_k$ such that $\alpha(B_k) = B'_k$ for all k ;
- (ii) for all positive integers n there exists a positive integer k such that $A_n \subseteq B'_k$.

PROOF. The proof for AF algebras without identity is the same as the one given in [1, 2.7, p. 208].

2.12. THEOREM. Each AF algebra has an associate diagram and each diagram is an associate diagram of some AF algebra; moreover, an associate diagram of an AF algebra A_1 is equivalent to an associate diagram of an AF algebra A_2 if and only if A_1 is *-isomorphic to A_2 .

PROOF. The first part is just a summary of the above. The second part follows from 2.8, 2.9, 2.10, and 2.11.

The next result relates the work in this section with Bratteli's original work.

2.13. PROPOSITION. Let A be an AF algebra and let $(D, d, \mathcal{U}) = ((i, m))_{m=1, i=1}^{\infty, r(m)}, d, \{U_m\}_{m=1}^{\infty})$ be an associate diagram of A . Then A has an identity if and only if there exists a positive integer n such that each $(i, m) \in D$, with $m \geq n$, has an ancestor and

$$d((j, m+1)) = \sum_{i=1}^{r(m)} \beta_{ij} d((i, m)),$$

where $1 \leq j \leq r(m+1)$, $m \geq n$, and β_{ij} denotes the entry of the i th row of the j th column of the matrix U_m .

PROOF. The proof is straightforward.

Now let (D, d, \mathcal{U}) be a diagram and let K be a subset of D that satisfies the following: if $x \in K$, then all descendants of x belong to K ; if $x \in D$ and all descendants of x at the next generation belong to K , then $x \in K$. The set K is called an *ideal subset* of D and if K is nonempty, the subdiagram $(K, d|_K, \mathcal{U}_K)$ generated by K is called an *ideal subdiagram* of (D, d, \mathcal{U}) . Note that the canonical ideal subdiagram associated with some ideal of a C*-algebra is an ideal subdiagram in the above sense. If $D \setminus K$ is nonempty and every two elements in $D \setminus K$ have a common descendant, then K is called a *primitive ideal subset* of D and $(K, d|_K, \mathcal{U}_K)$ is called a *primitive ideal subdiagram* of (D, d, \mathcal{U}) . A subdiagram $(E, d|_E, \mathcal{U}_E)$ of (D, d, \mathcal{U}) is called an *irreducible subdiagram* of (D, d, \mathcal{U}) if every two elements in E have a common descendant.

2.14. PROPOSITION. Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then the following statements are true:

- (i) If I is a closed two sided ideal of A , then the ideal I and the quotient space A/I are AF algebras.

(ii) If I is a nontrivial closed two sided ideal of A , then each associate diagram of I is equivalent to the canonical subdiagram $(K, d|_K, \mathcal{U}_K)$ of (D, d, \mathcal{U}) associated with I . Conversely, for each proper ideal subdiagram $(K, d|_K, \mathcal{U}_K)$ of (D, d, \mathcal{U}) there is a nontrivial closed two sided ideal I of A such that the canonical subdiagram of (D, d, \mathcal{U}) associated with I is $(K, d|_K, \mathcal{U}_K)$.

(iii) If I is a nontrivial closed two sided ideal of A and K is the ideal subset of D associated with I , then the subdiagram $(D \setminus K, d|_{D \setminus K}, \mathcal{U}_{D \setminus K})$ of (D, d, \mathcal{U}) is an associate diagram of the quotient C^* -algebra A/I .

(iv) If I is a nontrivial closed two sided ideal of A and K is the ideal subset of D associated with I , then I is a primitive ideal of A if and only if K is a primitive ideal subset of D .

(v) The AF algebra A is irreducible if and only if the associate diagram (D, d, \mathcal{U}) is irreducible.

PROOF. Bratteli's arguments found in [1] that prove the above assertions when A has an identity are valid even when A does not have an identity.

3. Approximately finite dimensional C^* -algebras and Bratteli diagrams. In this section we develop some new results that allow us to determine properties of an AF algebra by observing the characteristics of its diagram. In particular we characterize AF algebras that are liminal, postliminal, antiliminal, and with continuous trace; moreover, we characterize liminal AF algebras with Hausdorff spectrum.

3.1. DEFINITION. A diagram (D, d, \mathcal{U}) is called elementary if and only if (D, d, \mathcal{U}) is irreducible and has the following property: for each connected sequence $\{x_m\}_{m=1}^\infty$ in D there are natural numbers p and q such that, for all $m > p$, x_m is a descendant of x_1 with multiplicity q .

3.2. LEMMA. Let (D, d, \mathcal{U}) be a diagram. Then (D, d, \mathcal{U}) is elementary if and only if the following two conditions are satisfied:

(i) for each connected sequence $\{x_n\}$ in D there is a natural number p such that, for all $n > p$, x_{n+1} is a descendant of x_n with multiplicity one;

(ii) each pair of complete connected sequences $\{x_n\}$ and $\{y_n\}$ in D , with x_n and y_n at the same generation for all positive integers n , are eventually the same sequence.

PROOF. Suppose (D, d, \mathcal{U}) is elementary. Then condition (i) follows immediately. Now let $\{x_n\}$ and $\{y_n\}$ be complete connected sequences with x_n and y_n at the same generation. Without loss of generality we may assume that x_1, y_1 are at first generation. Next let z_2 be a common descendant of x_1, y_1 . Without loss of generality we may assume z_2 is at the second generation. Such a z_2 can be found since (D, d, \mathcal{U}) is irreducible. Next choose a z_3 that is a common descendant of x_2, y_2, z_2 . Again without loss of generality we may assume that z_3 is at the third generation. Continue on in this fashion and we can construct a complete connected sequence $\{z_n\}_{n=2}^\infty$ such that z_{n+1} is a common descendant of x_n, y_n, z_n at the $(n+1)$ th generation. But if $\{x_n\}$ and $\{y_n\}$ are not eventually the same, then either x_1, z_2, z_3, \dots or y_1, z_2, z_3, \dots contradicts our assumption about connected sequences in D . Thus (ii) holds.

Now assume (i) and (ii) hold. Clearly (ii) implies (D, d, \mathcal{U}) is irreducible. Let $\{x_n\}_{n=1}^\infty$ be a complete connected sequence and let r be a positive integer such that if $n \geq r$, then x_{n+1} is a descendant of x_n with multiplicity one. Without loss of generality we may assume that x_1 is at the first generation. Obviously, (i) and (ii) imply that there exists a $p \geq r$ such that for $n \geq p$, x_{n+1} is the only descendant of x_n at the $(n+1)$ th generation. Now let y_1, y_2, \dots, y_k be all of the descendants of x_1 at the p th generation different from x_p . We claim that there is a number $q > p$ such that for $n \geq q$, x_n is the only descendant of the set y_1, y_2, \dots, y_k at the n th generation. Once this claim is shown, it would follow that for $n \geq q$, x_n is the only descendant of x_1 at the n th generation. But clearly this would mean that (D, d, \mathcal{U}) is elementary and our proof would be complete.

Assume that there is a y_i from our set above that has infinitely many descendants not belonging in $\{x_n\}_{n=p+1}^\infty$; rename this element z_1 . Now the element z_1 must have a descendant $z_2 \neq x_{p+1}$ at the $(p+1)$ th generation and z_2 must have infinitely many descendants not in $\{x_n\}_{n=p+2}^\infty$. Similarly, z_2 must have a descendant $z_3 \neq x_{p+2}$ at the $(p+2)$ th generation and z_3 must have infinitely many descendants not in $\{x_n\}_{n=p+3}^\infty$. Thus one may construct inductively a complete connected sequence $\{z_k\}_{k=1}^\infty$ with z_k at the $(p+k-1)$ th generation and $z_k \neq x_{p+k-1}$ for $k = 1, 2, \dots$, which contradicts (ii). Hence our proof is complete.

The next result is a slight variant of 3.2.

3.3. COROLLARY. *The diagram (D, d, \mathcal{U}) is elementary if and only if (D, d, \mathcal{U}) is irreducible and for each complete connected sequence $\{x_n\}_{n=1}^\infty$ there is a positive integer p such that for all $n \geq p$, x_{n+1} is the only descendant of x_n at the generation of x_{n+1} , and it is a descendant of x_n with multiplicity one.*

3.4. LEMMA. *Let (D, d, \mathcal{U}) be an elementary diagram and let (E, e, \mathcal{V}) be a diagram equivalent to (D, d, \mathcal{U}) . Then (E, e, \mathcal{V}) is elementary.*

PROOF. The proof is straightforward.

3.5. LEMMA. *Let (D, d, \mathcal{U}) be an elementary diagram, $\{x_n\}_{n=1}^\infty$ a complete connected sequence in D with x_1 at the first generation and D_m the m th generation of D , for a fixed positive integer m . Then there exists a positive integer $p > m$ such that, for $n \geq p$, x_n is the only descendant of each element of D_m at the n th generation.*

PROOF. The proof is straightforward.

3.6. LEMMA. *Let $(D, d, \mathcal{U}) = (\{(i, m)\}_{m=1, i=1}^\infty, d, \{U_m\}_{m=1}^\infty)$ be an elementary diagram and let $\{x_n\}_{n=1}^\infty$ be a complete connected sequence in D . Then the subdiagram of (D, d, \mathcal{U}) generated by $\{x_n\}$ is equivalent to (D, d, \mathcal{U}) .*

PROOF. For each positive integer n let D_n denote the n th generation of D . We can, without loss of generality, assume $x_1 \in D_1$ and that each x_n is the first element in D_n ; moreover, by virtue of 3.5 and by dropping to a subdiagram of the first kind if necessary, we may assume that the only descendant of an element in D_n at the $(n+1)$ th generation is x_{n+1} . Now define a new diagram the following way: let $s(2n-1) = 1$ and $s(2n) = r(n)$, $n = 1, 2, \dots$, and set $E = \{(i, n)\}_{n=1, i=1}^\infty$; define

$e: E \rightarrow N$ by the formula

$$e((1, 2n - 1)) = d(x_n) \quad \text{and} \quad e((i, 2n)) = d((i, n)), \quad n = 1, 2, \dots;$$

for each integer n let V_{2n} be the $s(2n)$ by 1 matrix equal to the first column of U_n , and let V_{2n-1} be the 1 by $s(2n)$ matrix with 1 as the entry of the first column of the first row and 0 otherwise. Clearly $(E, e, \{V_n\})$ is a diagram; moreover, it is clear that (D, d, \mathcal{U}) , and the subdiagram of (D, d, \mathcal{U}) generated by $\{x_n\}$, are equivalent to subdiagrams of $(E, e, \{V_n\})$ of the first kind. Hence, by virtue of 2.6, our proof is complete.

3.7. COROLLARY. *Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then A is an elementary C^* -algebra if and only if the diagram (D, d, \mathcal{U}) is elementary.*

PROOF. Our assertion follows immediately from 2.12, 3.4, and 3.6.

3.8. THEOREM. *Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then A is liminal if and only if for each connected sequence $\{x_m\}_{m=1}^\infty$ in D there are natural numbers p and q such that for $m \geq p$, x_m is a descendant of x_1 with multiplicity q .*

PROOF. First assume that for each connected sequence $\{x_m\}_{m=1}^\infty$ in D there are natural numbers p and q such that for $m \geq p$, x_m is a descendant of x_1 with multiplicity q . Let I be a primitive ideal of A . By 2.14 there is a primitive ideal set $K \subseteq D$ associated with I such that $(E, d|E, \mathcal{U}_E)$ is an associate diagram of the quotient C^* -algebra A/I , where $E = D \setminus K$. Since $(E, d|E, \mathcal{U}_E)$ is irreducible, our assumption implies $(E, d|E, \mathcal{U}_E)$ is an elementary diagram. Thus by 3.7 A/I is elementary. It follows from [4, 4.1.10, p. 97] that A is liminal.

Now assume A is liminal and let $\{x_m\}_{m=1}^\infty$ be a connected sequence in D . Set $E = \{y \in D: y \text{ is an ancestor of some } x_m\}$. Clearly, the subdiagram $(E, d|E, \mathcal{U}_E)$ generated by E is irreducible; moreover, it is clear that $D \setminus E$ is a primitive ideal subset of D . It follows from 2.14 and 3.7 that $(E, d|E, \mathcal{U}_E)$ is elementary. Thus our proof is complete.

3.9. COROLLARY. *Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Let K be the set of all $x_1 \in D$ with the following property: for every connected sequence $\{x_m\}_{m=1}^\infty$ in D there are natural numbers p and q such that for all $m \geq p$, x_m is a descendant of x_1 with multiplicity q . Then K generates an ideal subdiagram of (D, d, \mathcal{U}) ; moreover, this ideal subdiagram is an associate diagram of the largest liminal ideal of A .*

PROOF. Clearly K generates an ideal subdiagram of (D, d, \mathcal{U}) and by virtue of 3.8 it is clear that it is an associate diagram of a liminal ideal of A . On the other hand, if I is the largest liminal ideal of A it has an associate ideal subset of L of D . Thus, by 3.8, $L \subseteq K$ and our proof is complete.

3.10. COROLLARY. *Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then A is antiliminal if and only if for each $x_1 \in D$ there exists a connected sequence $\{x_n\}_{n=1}^\infty$ such that, for each integer n , x_{n+1} is a descendant of x_n with multiplicity greater than one.*

PROOF. The proof follows immediately from 3.9.

3.11. THEOREM. *Let A be a liminal AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then the following statements are equivalent:*

- (i) *The spectrum \hat{A} of A is Hausdorff.*
- (ii) *Let $\{x_n\}$ and $\{y_n\}$ be complete connected sequences in D with x_n and y_n at the same generation for all positive integers n . If $\{x_n\}$ and $\{y_n\}$ are not eventually the same, then there exists a positive integer m such that x_m and y_m do not have a common descendant.*

PROOF. Suppose (ii) holds and \hat{A} is not Hausdorff. Let π_1, π_2 be points in \hat{A} , $\pi_1 \neq \pi_2$, that cannot be separated. Since A is liminal, \hat{A} is T_1 , so \hat{A} is homeomorphic to $\text{Prim}(A)$. Let $I_1 = \ker \pi_1$ and $I_2 = \ker \pi_2$, and let K_1 and K_2 be the primitive ideal subsets of D associated with I_1, I_2 , respectively. Let $\{x_n\}_{n=1}^\infty$ and $\{y_n\}_{n=1}^\infty$ be complete connected sequences in $D \setminus K_1$ and $D \setminus K_2$, respectively, with x_n and y_n at the same generation, and such that $\{x_n\}$ and $\{y_n\}$ are not eventually the same. Note that we are tacitly assuming that I_1 and I_2 are nontrivial, but this can be done without loss of generality. Let m be the number given by (ii) for the sequences $\{x_n\}$ and $\{y_n\}$. Let K_x be the set of all $z \in D$ such that all descendants of z at some generation are also descendants of x_m . Similarly define K_y . Note K_x and K_y are ideal subsets of D . Let I_x and I_y denote the closed two sided ideals of A that are associated with K_x and K_y , respectively. Let $W_x = \{I \in \text{Prim}(A): I_x \not\subseteq I\}$ and $W_y = \{I \in \text{Prim}(A): I_y \not\subseteq I\}$. Clearly, W_x and W_y are open neighborhoods of I_1 and I_2 , respectively. Since π_1 and π_2 cannot be separated in \hat{A} (and therefore I_1 and I_2 cannot be separated in $\text{Prim}(A)$), there is a primitive ideal $I \in W_x \cap W_y$. Let L be the ideal subset of D associated with I . Since $I_x \not\subseteq I$ and $I_y \not\subseteq I$, this implies $K_x \not\subseteq L$ and $K_y \not\subseteq L$. Hence $x_m \notin L$ and $y_m \notin L$. But L is a primitive ideal subset of D , so x_m and y_m have a common descendant, contradicting (ii). Therefore \hat{A} is Hausdorff.

Suppose \hat{A} is Hausdorff and $\{x_n\}$ and $\{y_n\}$ are complete connected sequences, with x_n and y_n at the same generation. Suppose $\{x_n\}$ and $\{y_n\}$ are not eventually the same. Let $E = \{z \in D: z \text{ is an ancestor of some } x_n\}$ and $F = \{z \in D: z \text{ is an ancestor of some } y_n\}$. Clearly, $D \setminus E$ and $D \setminus F$ are primitive ideal sets so $(E, d|_E, \mathcal{U}_E)$ and $(F, d|_F, \mathcal{U}_F)$ are elementary subdiagrams of (D, d, \mathcal{U}) . By virtue of 3.2 there is a positive integer p_1 such that $x_n \notin F$ and $y_n \notin E$ for $n > p_1$. Now assume that, for each integer $n > p_1$, x_n and y_n have a common descendant z_n . Let I_1 and I_2 be the primitive ideals of A associated with $(D \setminus E, d|(D \setminus E), \mathcal{U}_{D \setminus E})$ and $(D \setminus F, d|(D \setminus F), \mathcal{U}_{D \setminus F})$, respectively. Let J_1 and J_2 be closed two sided ideals of A so that $J_1 \not\subseteq I_1$ and $J_2 \not\subseteq I_2$. Let K_1 and K_2 denote the associated ideal subsets of D generated by J_1 and J_2 , respectively. It follows that $K_1 \not\subseteq D \setminus E$ and $K_2 \not\subseteq D \setminus F$. Thus $K_1 \cap E$ and $K_2 \cap F$ are nonempty. This means that we can find

a positive integer $p_2 > p_1$ such that for $n > p_2$, $z_n \in K_1 \cap K_2$. Next choose $n > p_2$ and then define the set L_n the following way: choose a complete connected sequence $\{w_i\}_{i=1}^\infty$ in D with $w_1 = z_n$; then let L_n be the set of all $w \in D$ such that the w is an ancestor of some w_i . Clearly, $D \setminus L_n$ is a primitive ideal set. Let I be the primitive ideal associated with $D \setminus L_n$. It is clear that $J_i \subseteq I$, $i = 1, 2$. Thus I_1 and I_2 cannot be separated. But this contradicts the fact that \hat{A} is Hausdorff. Hence there is an integer n for which x_n and y_n do not have a common descendant and therefore our proof is complete.

3.12. LEMMA. *Let A be an AF algebra and let $\{I_\rho\}_{0 < \rho < \alpha}$ be a composition sequence for A . If F is a finite dimensional factor contained in A , then there is an ordinal ρ , $0 < \rho < \alpha$, such that $F \not\subseteq I_\rho$ and $F \subseteq I_{\rho+1}$.*

PROOF. Let β be the smallest ordinal for which $F \subseteq I_\beta$. We shall show that β is not a limit ordinal. Suppose that it is; that is, suppose there is an increasing sequence $\{\beta_n\}$ of ordinals such that $\lim \beta_n = \beta$. Then $I_\beta = \overline{\bigcup I_{\beta_n}}$. Denote by p the unit of F . Clearly, there is a β_n and an $x \in I_{\beta_n}$ such that $\|p - x\| < 1$. Thus $\|p - pxp\| < 1$. Now let B denote the hereditary (order-related) C^* -subalgebra of A generated by p (see [9]). It is easy to see that $pxp \in B$ and that p is the identity of B . Therefore pxp is invertible in B . Since pxp belongs to I_{β_n} , p belongs to I_{β_n} . Hence $F \subseteq I_{\beta_n}$, a contradiction. So β is not a limit ordinal. Set $\rho = \beta - 1$ and our proof is complete.

3.13. THEOREM. *Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then the following statements are equivalent:*

- (i) A is postliminal.
- (ii) *For each connected sequence $\{x_n\}_{n=1}^\infty$ in D there is a positive integer n such that for $m \geq n$ the point x_{m+1} is a descendant of x_m with multiplicity one.*

PROOF. Assume (i) holds. Let $(I_\alpha)_{0 < \alpha < \rho}$ be a composition sequence for A such that each $I_{\alpha+1}/I_\alpha$ is liminal. For each ordinal α , $0 < \alpha < \rho$, let K_α denote the ideal subset of D associated with I_α . Suppose $\{x_m\}$ is a connected sequence with a subsequence $\{x_{m_i}\}$ with the property: x_{m_i+1} is a descendant of x_{m_i} with multiplicity greater than one, $i = 1, 2, \dots$. Without loss of generality we may assume x_{m+1} is a descendant of x_m with multiplicity greater than one, $m = 1, 2, \dots$. Let α_0 be the smallest ordinal for which K_{α_0} meets $\{x_m\}_{m=1}^\infty$ and let m_0 denote the smallest integer for which $x_{m_0} \in K_{\alpha_0}$. Clearly, α_0 is the smallest ordinal α for which $x_{m_0} \in K_\alpha$. Now x_{m_0} corresponds to a factor contained in A . By virtue of 3.12 it is easy to deduce that α_0 is not a limit ordinal. Set $\beta_0 = \alpha_0 - 1$. It follows that $\{x_m\}_{m=m_0}^\infty \subseteq K_{\beta_0+1} \setminus K_{\beta_0}$. But $K_{\beta_0+1} \setminus K_{\beta_0}$ generates a subdiagram of (D, d, \mathcal{U}) that is an associate diagram of the liminal AF algebra $I_{\beta_0+1}/I_{\beta_0}$. But this contradicts 3.8, so (ii) holds.

Now assume (ii) holds. Since (ii) still holds for every subdiagram of (D, d, \mathcal{U}) that is associated with a quotient C^* -algebra of A , it will suffice to show that A has a nonzero liminal two sided ideal.

We claim that there is an element x in D such that all descendants of x are with multiplicity one. Otherwise, let x_1 be any element in D and let x_2 be a descendant of x , with multiplicity greater than one. Continue on in this fashion and we construct a connected sequence $\{x_n\}_{n=1}^\infty$ such that x_{n+1} is a descendant of x_n with multiplicity greater than one. Since this violates (ii), the claim is established.

Let $x \in D$ be chosen so that all descendants of x are with multiplicity one. Let E be the subset of D that consists of x and all of its descendants. Then let L be the set of all $z \in D$ such that all descendants of z at some generation belong to E . It is easy to see that L is an ideal subset of D . Now let $\{y_n\}$ be a connected sequence in L . We wish to show that there are positive numbers p, q such that for all $n > p$, y_n is a descendant of y_1 with multiplicity q . Without loss of generality we can assume y_1 is at the first generation and, by dropping to a subdiagram of the first kind if necessary, we may assume $\{y_n\}$ is complete. Since $y_1 \in L$ there is a positive integer p such that all descendants of y_1 at the p th generation belong to E , in particular y_p belongs to E . It follows that for $n > p$, y_n is the only ancestor of y_{n+1} in E because y_{n+1} is a descendant of x with multiplicity one. Thus if y_p is a descendant of y_1 with multiplicity q , then so is y_n , $n > p$. It follows from 3.8 that the nontrivial closed two sided ideal of A associated with L is liminal and our proof is complete.

3.14. PROPOSITION. *Let A be an AF algebra and (D, d, \mathfrak{U}) an associate diagram of A . Then the following statements are equivalent:*

- (i) $\text{Prim}(A)$ is a T_1 topological space.
- (ii) *If $\{x_n\}$ and $\{y_n\}$ are two connected sequences that are contained in the complement of some primitive ideal subset of D , then for each positive integer m there exists a positive integer $n > m$ such that y_n is a descendant of x_m and x_n is a descendant of y_m .*

PROOF. Assume (i) holds and that $\{x_n\}$ and $\{y_n\}$ are two connected sequences that are contained in the complement of some primitive ideal subset K of D . Set $E = D \setminus K$. Assume (ii) is not true. Then we may assume that there exists a positive integer m_0 such that, for all $n > m_0$, y_n is not a descendant of x_{m_0} . Let $F = \{z \in D: z \text{ is the ancestor of some } y_n, n = 1, 2, \dots\}$. It is easy to see that $F \subseteq E$, $x_{m_0} \notin F$, and that $L = D \setminus F$ is a primitive ideal set. Let I and J be the primitive ideals associated with K and L , respectively. From the above and 2.14, we have that $I \subseteq J$ and $I \neq J$. Since $\text{Prim}(A)$ is a T_1 space, $\{I\}$ is closed. But by [4, 3.1.4, p. 70] I must be maximal among primitive ideals, which contradicts the fact that $I \neq J$. So (i) implies (ii).

Now assume (ii) holds and that $\text{Prim}(A)$ is not T_1 . By [4, 3.1.4, p. 70] there exist primitive ideals I and J such that $I \subseteq J$ and $I \neq J$. Let K and L be the primitive ideal sets associated with I and J , respectively. Let $E = D \setminus K$ and $F = D \setminus L$. Note $F \subseteq E$ and $F \neq E$. Let $x_1 \in E \setminus F$. Let $\{y_n\}_{n=1}^\infty$ be a connected sequence in F and let F_0 be the set of all $z \in D$ such that z is the ancestor of some y_n , $n = 1, 2, \dots$. Clearly, $F_0 \subseteq F$. Now let $\{x_n\}_{n=1}^\infty$ be any connected sequence in E . Note that no x_n can have a descendant in F_0 since $x_1 \notin F_0$. But this contradicts (ii), so $\text{Prim}(A)$ must be T_1 . This completes our proof.

Let $\{F_n\}_{n=1}^\infty$ be a sequence of factors with increasing dimension and for each positive integer n let ϕ_n be a $*$ -isomorphism of F_n into F_{n+1} . Let $A = \text{ind lim} \langle F_n, \phi_n \rangle$. Recall that there exists an increasing sequence of C^* -subalgebras $\{\tilde{F}_n\}$ of A (ordered by inclusion) which satisfies the following conditions: for each positive integer n there is a $*$ -isomorphism λ_n of F_n onto \tilde{F}_n such that $\lambda_{n+1}\phi_n = \lambda_n$; A is the uniform closure of $\cup \tilde{F}_n$.

3.15. LEMMA. Suppose $\{F_n\}$, $\{\phi_n\}$, A , $\{\tilde{F}_n\}$, $\{\lambda_n\}$ are defined as above. Then the following statements are equivalent:

- (i) If p is a minimal projection in F_n , then $\lambda_n(p)$ is a minimal projection in A .
- (ii) For each $m \geq n$, ϕ_m is a map with multiplicity one.

PROOF. The proof is straightforward.

3.16. THEOREM. Let A be an AF algebra and (D, d, \mathcal{U}) an associate diagram of A . Then the following statements are equivalent:

- (i) A is a C^* -algebra with continuous trace.
- (ii) For each connected sequence $\{x_n\}_{n=1}^\infty$ in D there is a positive integer m such that all the descendants of x_m are descendants with multiplicity one.

PROOF. Let $\{x_n\}_{n=1}^\infty$ be a connected sequence. Without loss of generality we may assume that $\{x_n\}_{n=1}^\infty$ is complete and x_1 belongs to the first generation. Let $E = \{x \in D: x \geq x_n \text{ for some } n \geq 1\}$. Note that E is the smallest set for which $\{x_n\} \subseteq E$ and $D \setminus E$ is a primitive ideal set. Let I_0 be the primitive ideal associated with $D \setminus E$. Let $K'_n = \{z \in D: z \leq x_n\}$ and then let K_n be the set of all $z \in D$ such that all descendants of z at some generation belong to K'_n . Clearly, K_n is an ideal subset of D . Let J_n be the closed two sided ideal of A associated with K_n . Next set

$$V_n = \{I \in \text{Prim}(A): J_n \not\subseteq I\}$$

and note V_n is an open neighborhood of I_0 . We claim $\{V_n\}$ forms a basis of open neighborhoods for I . Let W be an open neighborhood of I_0 . Then there is a closed two sided ideal I_1 of A such that $I_1 \not\subseteq I_0$ and $W = \{I \in \text{Prim}(A): I_1 \not\subseteq I\}$. Let L_1 be the ideal subset of D associated with I_1 . Since, by 2.14, $I_1 \not\subseteq I_0$ implies $L_1 \not\subseteq D \setminus E$, there exists an x in $L_1 \cap E$. But this means that there is an $x_n \in L_1$ for some positive integer n . Therefore, $K_n \subseteq L_1$, so $J_n \subseteq I_1$. This implies $V_n \subseteq W$. Hence $\{V_n\}$ forms a basis of open neighborhoods for I .

Now assume (i) holds. Let $\{x_n\}_{n=1}^\infty$ be a connected sequence in D . We can assume without loss of generality that x_1 is at the first generation and that $\{x_n\}_{n=1}^\infty$ is complete. Let

$$E = \{z \in D: z \geq x_n \text{ for some } x_n\}.$$

Note E is the smallest set that contains $\{x_n\}$ for which $D \setminus E$ is a primitive ideal set. Let π_0 be the irreducible representation associated with E , and let I_0 be the primitive ideal associated with $D \setminus E$. By virtue of [4, 4.5.3, p. 106], [2, p. 76], and [4, 10.5.6, p. 227] there exist an open compact neighborhood U of π_0 and a projection p in A such that $\pi(p)$ is a projection of rank 1 for all $\pi \in U$ and 0 for all $\pi \notin U$. Now let $\{A_n\}_{n=1}^\infty$, $\{F_{i,n}\}_{n=1}^\infty$, $\{F_{j,n}\}_{n=1}^\infty$, be such that A_n is a finite dimensional

C^* -subalgebra of A , $F_{i,n}$ is a factor contained in A_n , $A_n = \sum_{i=1}^{r(n)} F_{i,n}$, A is the uniform closure of $\cup A_n$, and (D, d, \mathcal{U}) is the diagram $\{A_n\}_{n=1}^\infty, \{F_{i,n}\}_{n=1, i=1}^\infty, r(n)$ generate. Since $\cup A_n$ is dense in A , it follows from [5, Lemma 1.6, p. 320] that there is a positive integer m_0 and a projection q in A_{m_0} such that $\|p - q\| < 1$. But this means $\pi(q)$ is a projection of rank 1 for all $\pi \in U$. Now let us view U as an open neighborhood of I_0 in $\text{Prim}(A)$. Let $\{V_n\}_{n=1}^\infty$ be the basis of open neighborhoods of I_0 defined as in the above paragraph. Next choose $m_1 > m_0$ such that $V_{m_1} \subseteq U$. Since $q \notin I_0$, there is an integer $m > m_1$ such that if $F_{i,m}$ is the factor in A_m that corresponds to x_m , then $q \cdot F_{i,m} \neq 0$. Let $q_i = q \cdot c_i \neq 0$, where c_i is the identity projection in $F_{i,m}$. Note $q_i + I$ is a projection of rank 1 in A/I for all $I \in V_m$. Now suppose $y \in D$, with $y < x_m$. Without loss of generality we may assume that y is at the next generation. Let $\{y_n\}$ be a complete connected sequence with $y_n = x_n$, $i = 1, 2, \dots, m, y_{m+1} = y$. Let $E_1 = \{z \in D: z \geq y_n \text{ for some } y_n\}$ and let I_1 be the primitive ideal associated with the primitive ideal set $D \setminus E_1$, which clearly belongs to V_m . So $q_i + I_1$ is a rank one projection in A/I_1 . It follows from 3.6 and 3.15 that y is a descendant of multiplicity one, so (ii) holds.

Assume (ii) holds. By virtue of 3.8 A is liminal and it follows from 3.11 that \hat{A} is Hausdorff. Let I_0 be a primitive ideal of A and let K be the associated primitive ideal set. Choose a complete connected sequence $\{x_n\} \subseteq D \setminus K$ with x_1 at the first generation. By (ii) there is a positive integer m for which all descendants of x_m are descendants with multiplicity one. Let q be a minimal projection in $F_{i,m}$, the factor contained in A associated with x_m . Now let $\{y_n\}$ be any complete connected sequence with $y_n = x_n$, $n = 1, 2, \dots, m$. Let $E = \{z \in D: z \geq y_n \text{ for some } y_n\}$. Recall that E is the smallest set that contains $\{y_n\}$ for which $D \setminus E$ is a primitive ideal set. Since A is liminal, the subdiagram of (D, d, \mathcal{U}) generated by E is elementary and therefore by 3.6 equivalent to the subdiagram of (D, d, \mathcal{U}) generated by $\{y_n\}_{n=m}^\infty$. By 3.15 $q + I$ is a projection of rank 1 in A/I , where I is the primitive ideal associated with $D \setminus E$. Now let $\{V_n\}$ denote a basis of open sets for I_0 defined as in the first paragraph of this proof with respect to $\{x_n\}$. We deduce from the above that $q + J$ is a projection of rank 1 in A/J , for all $J \in V_m$. Thus by [4, 4.5.3, p. 106], A is a C^* -algebra with continuous trace. Hence our proof is complete.

3.17. COROLLARY. *Let A be an AF algebra with continuous trace and (D, d, \mathcal{U}) an associate diagram of A . Then A is homogeneous of degree n , $1 < n < \aleph_0$, if and only if, for each connected sequence $\{x_k\}$, $\{d(x_k)\}$ converges to n .*

4. Examples. In this section we shall construct some examples of AF algebras with certain desired properties by utilizing the results of the previous sections. These examples demonstrate the power and usefulness of diagrams in the construction of examples.

In [6] Glimm proved that a separable C^* -algebra A is not postliminal if and only if there exists a sub- C^* -algebra B of A which has a UHF quotient. The next two examples show that this result cannot be improved when A is an AF algebra.

4.1. EXAMPLE. The AF algebra associated with the diagram below is an antiliminal C^* -algebra such that none of its quotients is UHF.

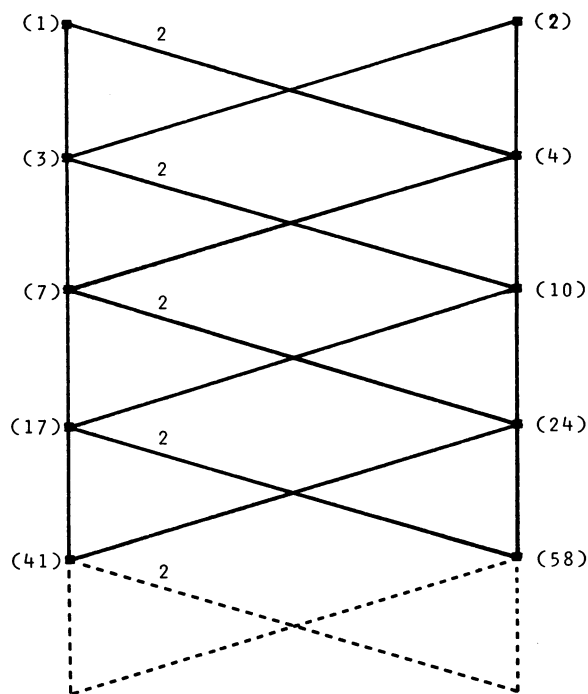


FIGURE 4

This example is due to Bratteli [1, 3.6, p. 213]. It is antiliminal by virtue of 3.10, it is clearly simple, and Bratteli proved that it is not $*$ -isomorphic to a UHF algebra [1, 3.6, p. 213].

4.2. EXAMPLE. The AF algebra associated with the diagram below is not a postliminal C^* -algebra, yet it does not contain any UHF subalgebras.

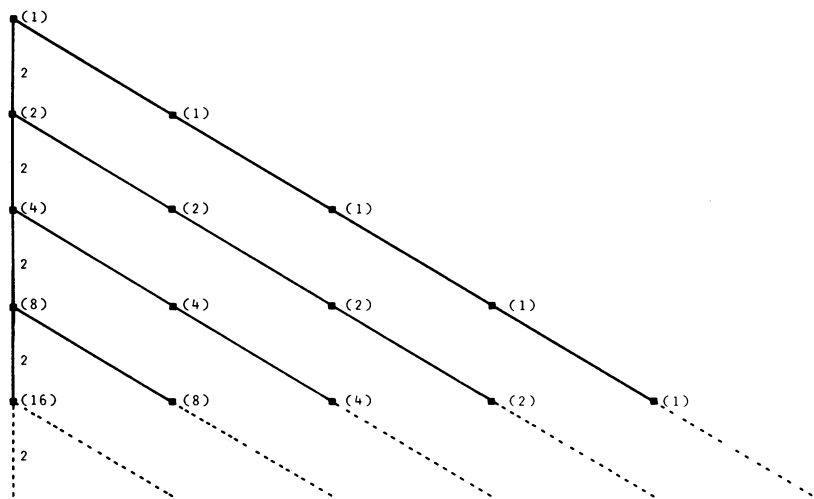


FIGURE 5

The C^* -algebra A associated with the diagram is not postliminal by virtue of 3.13. Now let $\{A_n\}_{n=1}^\infty$ be finite dimensional C^* -subalgebras of A , with $A_n \subseteq A_{n+1}$, such that the uniform closure of $\bigcup A_n$ is A . Next let $\{F_{i,n}\}_{n=1}^\infty$ be factors such that $F_{i,n} \subseteq A_n$ and $A_n = \sum_{i=1}^n F_{i,n}$. Finally, suppose the above diagram is the canonical diagram generated by $(\{A_n\}_{n=1}^\infty, \{F_{i,n}\}_{n=1}^\infty, \{i_n\}_{n=1}^\infty)$, where $i_n: A_n \rightarrow A_{n+1}$ is the identity map. Note $F_{i,n} \subseteq F_{1,n+1} \oplus F_{2,n+1}$. Suppose B is a UHF subalgebra with unit e . By [5, Lemmas 1.6, 1.8] there is a natural number n , a projection $p \in A_n$, and a partial isometry $u \in A$ such that $\|e - p\| < 1$, $u^*u = p$, and $uu^* = e$. Since

$$A_n = \sum_{i=1}^n \oplus F_{i,n} \quad \text{and} \quad F_{1,n} \subseteq F_{1,n+1} \oplus F_{2,n+1}, \quad p = p'_1 + p''_1 + \sum_{i=2}^n p_i$$

where

$$p'_1 \in F_{i,n+1}, \quad p''_1 \in F_{2,n+1}, \quad p_i \in F_{i,n}, \quad i = 2, 3, \dots, n,$$

are projections. Since B is UHF, we can choose an integer $m > 2^n$ such that there are m mutually equivalent orthogonal projections $\{e_i\}_{i=1}^m$ in B with $\sum_{i=1}^m e_i = e$. Clearly, $\{u^*e_iu\}_{i=1}^m$ are mutually equivalent orthogonal projections such that $\sum_{i=1}^m u^*e_iu = p$. Now let q'_1 be the identity for $F_{1,n+1}$, q''_1 the identity for $F_{2,n+1}$, and q_i the identity for $F_{i,n}$, $i = 2, 3, \dots, n$. It follows that for fixed j , $2 \leq j \leq n$, $\{u^*e_iuq_j\}_{i=1}^m$ are mutually equivalent orthogonal projections in $F_{j,n}$ with $\sum_{i=1}^m u^*e_iuq_j = p_j$. Since $m > 2^n > \dim F_{j,n}$, $p_j = 0$. Similarly, p'_1, p''_1 are zero. So $p = 0$. But this means $\|e\| = \|e - p\| < 1$, which is a contradiction. Hence A cannot contain any UHF subalgebras.

4.3. EXAMPLE. Let A be a postliminal C^* -algebra, and I the largest liminal closed two sided ideal of A . Dixmier observed in [3, Remarque C, p. 111] that each π in the spectrum \hat{I} of I has as its kernel a minimal primitive ideal of A . He then posed the following problem: Is there a minimal primitive ideal of A that is not the kernel of some $\pi \in \hat{I}$? This problem remained open for many years and was finally solved by Phillip Green [7]. By using our results on diagrams, we are able to construct below an elementary example of a postliminal C^* -algebra with the above property.

Let (D, d, \mathcal{Q}) be the diagram below. Clearly, $D = \{(i, n)\}_{n=1}^\infty$. The AF algebra A associated with (D, d, \mathcal{Q}) is postliminal by virtue of 3.13. The set $K = \{(2i-1, n): 1 \leq i \leq n-1, n = 2, 3, \dots\}$ is an ideal subset of D and, by virtue of 3.8, the closed two sided ideal I of A associated with K is the largest liminal ideal of A . Next let $L_1 = \{(i, n): 1 \leq i \leq 2n-2, n = 2, 3, \dots\}$. Clearly, L_1 is a primitive ideal set such that $K \subseteq L_1$, $K \neq L_1$, so the primitive ideal J_1 of A associated with L_1 contains I . It follows that the representation $\pi: A \rightarrow A/J_1$ does not belong to \hat{I} . We claim J_1 is a minimal primitive ideal of A . Suppose J_2 is a primitive ideal of A such that $J_2 \subseteq J_1$, $J_2 \neq J_1$, and let L_2 be the primitive ideal set associated with J_2 . Since $L_2 \subseteq L_1$, $L_2 \neq L_1$, we can choose $(i, n) \in L_1 \setminus L_2$, thus, $1 \leq i \leq 2n-2$, $n \geq 2$. But $(2n-1, n)$ does not belong to L_2 either and (i, n) and $(2n-1, n)$ do not have a common descendant. This contradicts the fact that L_2 is a primitive ideal set, so J_1 is minimal.

D is $K_{i,n}$. Clearly, $I_{i,n}$ is primitive and $A/I_{i,n}$ is a finite dimensional factor. Let $\pi_{i,n}$ be an irreducible representation whose kernel is $I_{i,n}$. Let $F_{i,n}$ be the finite dimensional factor associated with the point $(i, n) \in D$. It is clear that $\pi_{i,n}|_{F_{i,n}}$ is a *-isomorphism. Due to the fact that $\bigcup_{n=1}^{\infty} \sum_{i=1}^{2^n-1} \oplus F_{i,n}$ is dense in A , it follows that $\bigcap_{(i,n)} \pi_{i,n}^{-1}(0) = \{0\}$.

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